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# RATIONAL APPROXIMATION OF AFFINE COORDINATE SUBSPACES OF EUCLIDEAN SPACE

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Abstract. We show that affine coordinate subspaces of dimension at least two in Euclidean space are of Khintchine type for divergence. For affine coordinate subspaces of dimension one, we prove a result which depends on the dual Diophantine type of the basepoint of the subspace. These results provide evidence for the conjecture that all affine subspaces of Euclidean space are of Khintchine type for divergence.

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## 1. Introduction and results

The field of *simultaneous Diophantine approximation* is concerned with how well points in $\mathbb{R}^d$ can be approximated by points in $\mathbb{Q}^d$. One way in which the quality of approximation is measured is via some function $\psi : \mathbb{N} \to \mathbb{R}^+ \cup \{0\}$ which we often think of as a rate of approximation. Specifically, a vector $\boldsymbol{x} \in \mathbb{R}^d$ is said to be *$\psi$-approximable* if there are infinitely many $q \in \mathbb{N}$ such that $\|q\boldsymbol{x}\| < \psi(q)$, where $\|\cdot\|$ denotes sup-norm distance to the nearest integer point. Given $\psi$, it is natural to wonder about the size of the set of all $\psi$-approximable vectors. *Khintchine's theorem* [Khi26] answers this question when 'size' means Lebesgue measure. It is the foundational result of *metric Diophantine approximation*, and it states that if $\psi$ is nonincreasing, then the set of $\psi$-approximable vectors in $\mathbb{R}^d$ has either zero measure or full measure, depending on whether the series $\sum_{q\in\mathbb{N}} \psi(q)^d$ converges or diverges, respectively. Gallagher [Gal65] proved that if $d \geq 2$ then one can remove the monotonicity assumption on $\psi$.

While Khintchine's theorem gives us a means to determine the measure of $\psi$-approximable vectors, it leaves us completely in the dark regarding approximation within sets of zero measure. Such problems arise very naturally. For instance, in two dimensions, Khintchine's theorem tells us that if $\sum_{q\in\mathbb{N}} \psi(q)^2$ diverges, then for almost every $x \in \mathbb{R}$ the set $\{y \in \mathbb{R} : (x, y) \text{ is } \psi\text{-approximable}\} \subseteq \mathbb{R}$ has full Lebesgue measure. However, *a priori*, any given value of $x$, for example $x = \sqrt{2}$, may be an exception to this almost everywhere statement. Ideally, we would like to obtain a statement of the following form: Let $\ell$ and $k$ be positive integers with $\ell + k = d$, and let $\psi$ be a nonincreasing approximating function. Then, for $\boldsymbol{x} \in \mathbb{R}^\ell$, the set of $\psi$-approximable vectors in $\{\boldsymbol{x}\} \times \mathbb{R}^k \subseteq \mathbb{R}^d$ has either zero or full $k$-dimensional Lebesgue measure, depending on whether the sum $\sum_{q\in\mathbb{N}} \psi(q)^d$ converges or diverges, respectively.

However, upon choosing a rational vector $\boldsymbol{x}$, it is easily established that the convergence part of the above statement cannot be true. Hence, it is worth treating the different sides of the problem separately. The convergence side has previously been addressed in greater generality by Ghosh [Gho05] (and will not be visited in this paper). There, he shows that whether an affine subspace enjoys the convergence property depends on the Diophantine type of the matrix used to define the subspace.

An affine coordinate subspace $\{\boldsymbol{x}\}\times\mathbb{R}^k \subseteq \mathbb{R}^d$ is said to be of *Khintchine type for divergence* if for any nonincreasing function $\psi : \mathbb{N} \to \mathbb{R}^+$ such that $\sum_{q\in\mathbb{N}} \psi(q)^d$ diverges, almost every point on $\{\boldsymbol{x}\}\times\mathbb{R}^k$ is $\psi$-approximable. Intuitively, $\{\boldsymbol{x}\}\times\mathbb{R}^k$ is of Khintchine type for divergence if its typical points behave like the typical points of Lebesgue measure with respect to the divergence case of Khintchine's theorem. The recent article [Ram15] addresses the issue for certain affine coordinate hyperplanes in $\mathbb{R}^d$, where $d \geq 3$. There, sufficient conditions are given for a hyperplane to be of Khintchine type for divergence. In this note we settle the case of affine coordinate subspaces of dimension at least two and make vast progress on the one-dimensional case.

**Remark.** It is worth noting that the notions of Khintchine types for convergence and divergence can be analogously defined for general manifolds. It has been proved by Beresnevich–Dickinson–Velani [BDV07] and Beresnevich [Ber12] that analytic nondegenerate submanifolds of $\mathbb{R}^d$ are of Khintchine type for divergence. Here, an analytic submanifold of $\mathbb{R}^d$ is said to be *nondegenerate* if it is not contained in any affine hyperplane.

In fact, it is conjectured that nondegenerate submanifolds of $\mathbb{R}^d$ are also of Khintchine type for convergence, and this is known in several cases [BVVZ, Sim]. This contrasts with the situation for degenerate manifolds (e.g. affine subspaces), as the aforementioned results of Ghosh indicate. So it is interesting that in the divergence case, it does not seem to matter whether a manifold is degenerate or not.

Coming back to the present problem, we prove the following:

**Theorem 1.** *Every affine coordinate subspace of Euclidean space of dimension at least two is of Khintchine type for divergence.*

**Remark.** Combining Theorem 1 with Fubini's theorem shows that every submanifold of Euclidean space which is foliated by affine coordinate subspaces of dimension at least two is of Khintchine type for divergence. For example, given $a, b, c \in \mathbb{R}$ with $(a, b) \neq (0, 0)$, the three-dimensional affine subspace

$$\{(x, y, z, w) : ax + by = c\} \subseteq \mathbb{R}^4$$

is of Khintchine type for divergence, being foliated by the two-dimensional affine coordinate subspaces $(x, y) \times \mathbb{R}^2$ $(x, y \in \mathbb{R},\ ax + by = c)$.

The reason for the restriction to subspaces of dimension at least two is that Gallagher's theorem is used in the proof, and it is only true in dimensions at least two. Regarding one-dimensional affine coordinate subspaces, we have the following weaker theorem:

**Theorem 2.** *Consider a one-dimensional affine coordinate subspace $\{\boldsymbol{x}\} \times \mathbb{R} \subseteq \mathbb{R}^d$, where $\boldsymbol{x} \in \mathbb{R}^{d-1}$.*

(i) *If the dual Diophantine type of $\boldsymbol{x}$ is strictly greater than $d$, then $\{\boldsymbol{x}\}\times\mathbb{R}$ is contained in the set of very well approximable vectors*

$$\mathrm{VWA}_d = \{\boldsymbol{y} : \exists\varepsilon > 0\ \exists^\infty q \in \mathbb{N}\ \ \|q\boldsymbol{y}\| < q^{-1/d-\varepsilon}\}.$$

(ii) *If the dual Diophantine type of $\boldsymbol{x}$ is strictly less than $d$, then $\{\boldsymbol{x}\}\times\mathbb{R}$ is of Khintchine type for divergence.*

Here the *dual Diophantine type* of a point $\boldsymbol{x}\in\mathbb{R}^\ell$ is the number

$$\sup\left\{\tau\in\mathbb{R}^+ : \|\langle \boldsymbol{n},\boldsymbol{x}\rangle\| < |\boldsymbol{n}|_\infty^{-\tau} \text{ for i.m. } \boldsymbol{n}\in\mathbb{Z}^\ell\setminus\{\boldsymbol{0}\}\right\}. \tag{1}$$

**Remark.** The inclusion $\{\boldsymbol{x}\}\times\mathbb{R}\subseteq \mathrm{VWA}_d$ in part (i) is philosophically "almost as good" as being of Khintchine type for divergence, since it implies that for sufficiently "nice" functions $\psi:\mathbb{N}\to\mathbb{R}^+$ such that $\sum_{q\in\mathbb{N}}\psi(q)^d$ diverges, almost every point on $\{\boldsymbol{x}\}\times\mathbb{R}$ is $\psi$-approximable. For example, call a function $\psi$ *good* if for each $c>0$, we have either $\psi(q)\geq q^{-c}$ for all $q$ sufficiently large or $\psi(q)\leq q^{-c}$ for all $q$ sufficiently large. Then by the comparison test, if $\psi$ is a good function such that $\sum_{q\in\mathbb{N}}\psi(q)^d$ diverges, then for all $\varepsilon>0$, we have $\psi(q)\geq q^{-1/d-\varepsilon}$ for all $q$ sufficiently large, and thus by Theorem 2(i), every point of $\{\boldsymbol{x}\}\times\mathbb{R}$ is $\psi$-approximable. The class of good functions includes the class of *Hardy L-functions* (those that can be written using the symbols $+,-,\times,\div,\exp$, and $\log$ together with the constants and the identity function) [Har71, Chapter III]; *cf.* [AvdD05] for further discussion and examples.

Taken together, parts (i) and (ii) of Theorem 2 imply that if $\psi$ is a Hardy $L$-function such that $\sum_{q\in\mathbb{N}}\psi(q)^d$ diverges, and if $\boldsymbol{x}\in\mathbb{R}$ is a vector whose dual Diophantine type is not exactly equal to $d$, then almost every point of $\{\boldsymbol{x}\}\times\mathbb{R}\subseteq\mathbb{R}^d$ is $\psi$-approximable. This situation is somewhat frustrating, since it seems strange that points in $\mathbb{R}^{d-1}$ with dual Diophantine type exactly equal to $d$ should have any special properties (as opposed to those with dual Diophantine type $(d-1)$, which are the "not very well approximable" points). However, it seems to be impossible to handle these points using our techniques.

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## 2. Proof of Theorem 1: Subspaces of dimension at least two

Consider an affine coordinate subspace $\{\boldsymbol{x}\}\times\mathbb{R}^k$, where $\boldsymbol{x}\in\mathbb{R}^\ell$ and $\ell+k=d$. Given a nonincreasing function $\psi:\mathbb{N}\to\mathbb{R}^+$, for each $M,N$ with $M<N$ let

$$Q(M,N) := Q_{\boldsymbol{x},\psi}(M,N) = |\{M<q\leq N : \|q\boldsymbol{x}\|<\psi(N)\}|,$$

and write $Q(N):=Q(0,N)$. Since any real number $\delta>0$ may be thought of as a constant function, the expression $Q_\delta(M,N)$ makes sense. Note that $Q_\psi(M,N)=Q_{\psi(N)}(M,N)$.

**Lemma 3.** *For all $N\in\mathbb{N}$,*

$$Q_\delta(N) = |\{q\in\mathbb{N} : \|q\boldsymbol{x}\|<\delta, q\leq N\}| \geq N\delta^\ell - 1.$$

*Proof.* Let $\mathcal{Q}_\delta(N)=\{q\in\mathbb{N} : \|q\boldsymbol{x}\|<\delta, q\leq N\}$, so that $Q_\delta = |\mathcal{Q}_\delta|$.

We first claim that $Q_\delta(N)\geq Q_{\frac{\delta}{2},\boldsymbol{\gamma}}(N)-1$ for any $\boldsymbol{\gamma}\in\mathbb{R}^\ell$ and $N\in\mathbb{N}$, where

$$\mathcal{Q}_{\delta,\boldsymbol{\gamma}} := \{q\in\mathbb{N} : \|q\boldsymbol{x}+\boldsymbol{\gamma}\|<\delta, q\leq N\}$$

and $Q_{\delta,\boldsymbol{\gamma}}(N) = |\mathcal{Q}_{\delta,\boldsymbol{\gamma}}(N)|$. Simply notice that if $q_1 < q_2 \in \mathcal{Q}_{\frac{\delta}{2},\boldsymbol{\gamma}}(N)$, then by the triangle inequality, $q_2 - q_1 \in \mathcal{Q}_\delta(N)$. Therefore, letting $q_0 = \min \mathcal{Q}_{\frac{\delta}{2},\boldsymbol{\gamma}}(N)$, we have $\mathcal{Q}_{\frac{\delta}{2},\boldsymbol{\gamma}}(N) - q_0 \subseteq \mathcal{Q}_\delta(N) \cup \{0\}$, which implies that $Q_\delta(N) \geq Q_{\delta/2,\boldsymbol{\gamma}}(N) - 1$.

Now we show that for any $N \in \mathbb{N}$ there is some $\boldsymbol{\gamma}$ such that $Q_{\frac{\delta}{2},\boldsymbol{\gamma}}(N) \geq N\delta^\ell$. Notice that

$$\int_{\mathbb{T}^\ell} Q_{\frac{\delta}{2},\boldsymbol{\gamma}}(N)\, d\boldsymbol{\gamma} = \int_{\mathbb{T}^\ell} \sum_{q=1}^{N} \mathbf{1}_{(-\frac{\delta}{2},\frac{\delta}{2})^\ell}(q\boldsymbol{x} + \boldsymbol{\gamma})\, d\boldsymbol{\gamma} = N\delta^\ell.$$

Therefore, $Q_{\frac{\delta}{2},\boldsymbol{\gamma}}(N)$ must take some value $\geq N\delta^\ell$ at some $\boldsymbol{\gamma}$. Combining this with the previous paragraph proves the lemma. $\square$

**Lemma 4.** *Suppose that $\sum_{q\in\mathbb{N}} \psi(q)^d$ diverges. Then*

$$\sum_{\|q\boldsymbol{x}\| < \psi(q)} \psi(q)^k = \infty. \tag{2}$$

*Proof.* We may assume without loss of generality that $\psi(q) = 2^{-m_q}$ where $m_q \in \mathbb{N}$. (Indeed, given any $\psi$ as in the theorem statement, we can let $m_q = \lceil -\log_2 \psi(q) \rceil$ and replace $\psi(q)$ with $2^{-m_q}$. We will have changed $\psi$ by no more than a factor of $\frac{1}{2}$, preserving the divergence of the series $\sum_{q\in\mathbb{N}} \psi(q)^d$, but since the new $\psi$ is less than the old $\psi$, divergence of (2) for the new $\psi$ implies divergence of (2) for the old $\psi$.) Now,

$$\begin{aligned}
\sum_{\|q\boldsymbol{x}\| < \psi(q)} \psi(q)^k &\geq \sum_{m\in\mathbb{N}} \psi(2^m)^k \left|\left\{2^{m-1} < q \leq 2^m : \|q\boldsymbol{x}\| < \psi(2^m)\right\}\right| \\
&= \sum_{m\in\mathbb{N}} \psi(2^m)^k Q(2^{m-1}, 2^m) \\
&= \sum_{m\in\mathbb{N}} \sum_{n\geq m} \left(\psi(2^n)^k - \psi(2^{n+1})^k\right) Q(2^{m-1}, 2^m) \\
&= \sum_{n\in\mathbb{N}} \left(\psi(2^n)^k - \psi(2^{n+1})^k\right) \sum_{m=1}^{n} Q(2^{m-1}, 2^m) \\
&\geq \sum_{n\in\mathbb{N}} \left(\psi(2^n)^k - \psi(2^{n+1})^k\right) Q(2^n) \\
&\geq \sum_{n\in\mathbb{N}} \left(\psi(2^n)^k - \psi(2^{n+1})^k\right) \left[2^n \psi(2^n)^\ell - 1\right] \quad \text{(Lemma 3)} \\
&= -\psi(1)^k + \sum_{n\in\mathbb{N}} \left(\psi(2^n)^k - \psi(2^{n+1})^k\right) 2^n \psi(2^n)^\ell.
\end{aligned}$$

Now, let $(n_j)_{j=1}^\infty$ be the sequence indexing the set $\{n \in \mathbb{N} : m_{2^n} \neq m_{2^{n+1}}\}$ in increasing order. Then we have $\psi(2^{n_j})^k - \psi(2^{n_j+1})^k \gg \psi(2^{n_j})^k$, where $\gg$ is the Vinogradov symbol. So

$$\begin{aligned}\sum_{n\in\mathbb{N}} \left(\psi(2^n)^k - \psi(2^{n+1})^k\right) 2^n \psi(2^n)^\ell &\gg \sum_{j\in\mathbb{N}} 2^{n_j}\psi(2^{n_j})^{k+\ell} \\ &\gg \sum_{j\in\mathbb{N}} \left(\sum_{m=n_{j-1}+1}^{n_j} 2^m\right) \psi(2^{n_j})^d \\ &= \sum_{m\in\mathbb{N}} 2^m \psi(2^m)^d,\end{aligned}$$

which diverges by Cauchy's condensation test. □

*Proof of Theorem 1.* Suppose that $k \geq 2$. Then by Lemma 4, we can apply Gallagher's extension of Khintchine's theorem [Gal65] to the function

$$\psi_{\boldsymbol{x}}(q) = \begin{cases} \psi(q) & \|q\boldsymbol{x}\| < \psi(q) \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

and get that $\{\boldsymbol{x}\} \times \mathbb{R}^k$ is of Khintchine type for divergence. But $\boldsymbol{x} \in \mathbb{R}^\ell$ was arbitrary, and applying permutation matrices does not affect whether a manifold is of Khintchine type for divergence. This completes the proof. □

## 3. Proof of Theorem 2(i): Base points of high Diophantine type

The proof of Theorem 2(i) is based on the following standard fact, which can be found for example in [Cas57, Theorem V.IV]:

**Khintchine's transference principle.** *Let $\boldsymbol{x} \in \mathbb{R}^d$ and define the numbers*

$$\omega_D := \omega_D(\boldsymbol{x}) = \sup\left\{\omega \in \mathbb{R}^+ : \|\langle \boldsymbol{n}, \boldsymbol{x}\rangle\| \leq |\boldsymbol{n}|_\infty^{-(d+\omega)} \text{ for i.m. } \boldsymbol{n} \in \mathbb{Z}^d \backslash \{0\}\right\}.$$

*and*

$$\omega_S := \omega_S(\boldsymbol{x}) = \sup\left\{\omega \in \mathbb{R}^+ : \|q\boldsymbol{x}\| \leq q^{-(1+\omega)/d} \text{ for i.m. } q \in \mathbb{N}\right\}.$$

*Then*

$$\frac{\omega_D}{d^2 + (d-1)\omega_D} \leq \omega_S \leq \omega_D$$

*where the cases $\omega_D = \infty$ and $\omega_S = \infty$ should be interpreted in the obvious way.*

Note that $\omega_D$ is related to the dual Diophantine type $\tau_D$ defined in (1) via the formula $\tau_D(\boldsymbol{x}) = \omega_D(\boldsymbol{x}) + d$.

*Proof of Theorem 2(i).* We fix $\boldsymbol{x} = (x_1, \ldots, x_{d-1}) \in \mathbb{R}^{d-1}$ such that $\tau_D(\boldsymbol{x}) > d$, and we consider a point $(\boldsymbol{x}, y) \in \{\boldsymbol{x}\} \times \mathbb{R}$. It is clear from (1) that $\tau_D(\boldsymbol{x}, y) \geq \tau_D(\boldsymbol{x})$, so $\tau_D(\boldsymbol{x}, y) > d$ and thus $\omega_D(\boldsymbol{x}, y) > 0$. Thus by Khintchine's transference principle, $\omega_S(\boldsymbol{x}, y) > 0$, *i.e.* $(\boldsymbol{x}, y) \in \mathrm{VWA}_d$. □

## 4. Proof of Theorem 2(ii): Base points of low Diophantine type

Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be nonincreasing and such that $\sum_{q\in\mathbb{N}} \psi(q)^d$ diverges. Our goal here is to use the ideas of ubiquity theory to show that almost every point on $\{\boldsymbol{x}\} \times \mathbb{R} \subseteq \mathbb{R}^d$ is $\psi$-approximable, where $\boldsymbol{x} \in \mathbb{R}^{d-1}$ has been fixed with dual Diophantine type strictly less than $d$. The ubiquity approach begins with the fact that for any $N \in \mathbb{N}$ such that

$$N^{-1/(d-1)} < \psi(N) < 1, \tag{4}$$

we have

$$[0,1] \subseteq \bigcup_{\substack{q\leq N \\ \|q\boldsymbol{x}\|<\psi(N)}} \bigcup_{p=0}^{q} B\left(\frac{p}{q}, \frac{2}{qN\psi(N)^{d-1}}\right), \tag{5}$$

which is a simple consequence of Minkowski's theorem. The basic aim is to show that a significant proportion of the measure of the above double-union set is represented by $q$s that are closer to $N$ than to 0. Specifically, we must show that for some $k \geq 2$, the following three things hold:

**(U)** The pair $(\mathcal{R}, \beta)$ forms a (global) **ubiquitous system** with respect to the triple $(\rho, l, u)$, where

$$\begin{aligned} J &= \{p/q \in \mathbb{Q} : \|q\boldsymbol{x}\| < \psi(q)\}, & R_{p/q} &= \{p/q\} \quad (p/q \in J), \\ \mathcal{R} &= \{R_{p/q} : p/q \in J\}, & \beta_{p/q} &= q \quad (p/q \in J), \\ l_j &= k^{j-1} \quad (j \in \mathbb{N}), & u_j &= k^j \quad (j \in \mathbb{N}), \\ \rho(q) &= \frac{c}{q^2\psi(q)^{d-1}} \quad (q \in \mathbb{N}), \end{aligned}$$

and $c > 0$ will be chosen later. This means that there is some $\kappa > 0$ such that

$$\lambda\left([0,1] \cap \bigcup_{\substack{k^{j-1}<q\leq k^j \\ \|q\boldsymbol{x}\|<\psi(k^j)}} \bigcup_{p=0}^{q} B\left(\frac{p}{q}, \frac{c}{k^{2j}\psi(k^j)^{d-1}}\right)\right) \geq \kappa$$

for all $j$ sufficiently large.

**(R)** The function $\Psi(q) := \psi(q)/q$ is $u$**-regular**, meaning that there is some constant $\kappa < 1$ such that $\Psi(k^{j+1}) \leq \kappa\Psi(k^j)$ for all $j$ sufficiently large.

**(D)** The sum $\sum_{j\in\mathbb{N}} \frac{\Psi(k^j)}{\rho(k^j)}$ **diverges**.

Then [BDV06, Corollary 2] will imply that the set of $\psi_{\boldsymbol{x}}$-approximable numbers (see (3)) in $\mathbb{R}$ has positive measure, and Cassels' "0-1 law" [Cas50] will imply that it has *full* measure. Since the set of $\psi_{\boldsymbol{x}}$-approximable numbers is just the set of $y \in \mathbb{R}$ for which $(\boldsymbol{x}, y)$ is $\psi$-approximable, this will show that the set of $\psi$-approximable points on the line $\{\boldsymbol{x}\} \times \mathbb{R} \subseteq \mathbb{R}^d$ has full (one-dimensional Lebesgue) measure.

The following lemma shows that **(R)** and **(D)** are easy.

**Lemma 5.** *If $\psi : \mathbb{N} \to \mathbb{R}^+$ is nonincreasing, then* **(R)** *holds. Furthermore, if $\sum_{q\in\mathbb{N}} \psi(q)^d$ diverges, then* **(D)** *holds.*

*Proof.* In the first place, we have

$$\frac{\Psi(k^{j+1})}{\Psi(k^j)} = \frac{\psi(k^{j+1})}{k\psi(k^j)} \le \frac{1}{k},$$

which proves **(R)**. For **(D)**,

$$\sum_{j\in\mathbb{N}} \frac{\Psi(k^j)}{\rho(k^j)} = \sum_{j\in\mathbb{N}} k^j\psi(k^j)^d,$$

which diverges by Cauchy's condensation test. □

The challenge then is to establish **(U)**.

**Lemma 6.** *Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be nonincreasing such that (4) holds for all sufficiently large $N$, and assume that for all $k \ge 2$, there exists $j_k \ge 1$ such that for all $j \ge j_k$,*

$$\left|\left\{0 < q \le k^{j-1} : \|q\boldsymbol{x}\| < \psi(k^j)\right\}\right| \ll k^{j-1}\psi(k^j)^{d-1},$$

*where $\ll$ is the Vinogradov symbol, whose implied constant is assumed to be independent of $k$. Then* **(U)** *holds for some $k \ge 2$.*

*Proof.* For all $k \ge 2$ and $j \ge j_k$, we have

$$\lambda\left([0,1] \cap \bigcup_{\substack{q\le k^{j-1}\\ \|q\boldsymbol{x}\|<\psi(k^j)}} \bigcup_{p=0}^{q} B\left(\frac{p}{q}, \frac{2}{qk^j\psi(k^j)^{d-1}}\right)\right) \le \sum_{\substack{q\le k^{j-1}\\ \|q\boldsymbol{x}\|<\psi(k^j)}} \frac{4}{k^j\psi(k^j)^{d-1}} \ll \frac{1}{k}.$$

After choosing $k$ to be larger than the implied constant in the "$\ll$" comparison, we see that the left hand side is $\le 1-\kappa < 1$ for some $\kappa > 0$.

Combining with (5), we see that for all $j \ge j_k$ large enough so that (4) holds for $N = k^j$, we have

$$\lambda\left([0,1] \cap \bigcup_{\substack{k^{j-1}<q\le k^j\\ \|q\boldsymbol{x}\|<\psi(k^j)}} \bigcup_{p=0}^{q} B\left(\frac{p}{q}, \frac{2}{qk^j\psi(k^j)^{d-1}}\right)\right) \ge \kappa > 0,$$

and this implies **(U)** with $c = 2k$. □

The one-dimensional case of the following lemma was originally proven by Beresnevich, Haynes, and Velani using a continued fraction argument. This argument will appear in a forthcoming paper of theirs, currently in preparation [BHV].

**Lemma 7.** *Fix $\boldsymbol{x} \in \mathbb{R}^\ell$ and $\tau > \tau_D(\boldsymbol{x})$. Then for all $N$ sufficiently large and for all $\delta \ge N^{-1/\tau}$, we have*

$$|\{q \in \mathbb{N} : \|q\boldsymbol{x}\| < \delta, q \le N\}| \le 4^{\ell+1}N\delta^\ell. \tag{6}$$

*Proof.* Consider the lattice $\Lambda = g_t u_{\boldsymbol{x}}\mathbb{Z}^{\ell+1}$, where

$$g_t = \begin{bmatrix} e^{t/\ell}I_\ell & \\ & e^{-t} \end{bmatrix}$$

$$u_{\boldsymbol{x}} = \begin{bmatrix} I_\ell & -\boldsymbol{x} \\ & 1 \end{bmatrix},$$

where $t$ is chosen so that $R := e^{t/\ell}\delta = e^{-t}N$, *i.e.* $t = \log(N/\delta)/(1+1/\ell)$. Then (6) can be rewritten as

$$|\{\boldsymbol{r} \in \Lambda : |\boldsymbol{r}|_\infty < R\}| \le (4R)^{\ell+1}.$$

Now let $\mathcal{D}$ be the Dirichlet fundamental domain for $\Lambda$ centered at $\mathbf{0}$, *i.e.*

$$\mathcal{D} = \{\boldsymbol{r} \in \mathbb{R}^{\ell+1} : \mathrm{dist}(\boldsymbol{r}, \Lambda) = \mathrm{dist}(\boldsymbol{r}, \mathbf{0}) = |\boldsymbol{r}|_\infty\}.$$

Since $\Lambda$ is unimodular, $\mathcal{D}$ is of volume 1, so

$$|\{\boldsymbol{r} \in \Lambda : |\boldsymbol{r}|_\infty < R\}| = \lambda\left(\bigcup_{\substack{\boldsymbol{r}\in\Lambda \\ |\boldsymbol{r}|_\infty < R}} (\boldsymbol{r} + \mathcal{D})\right) \le^* \lambda(B_{\ell+1}(\mathbf{0}, 2R)) = (4R)^{\ell+1},$$

where the starred inequality is true as long as $\mathcal{D} \subseteq B_{\ell+1}(\mathbf{0}, R)$. So we need to show that $\mathcal{D} \subseteq B_{\ell+1}(\mathbf{0}, R)$ assuming that $N$ is large enough.

Suppose that $\mathcal{D} \not\subseteq B_{\ell+1}(\mathbf{0}, R)$. Then the last successive minimum of $\Lambda$ is $\gg R$, so by [Cas97, Theorem VIII.5.VI], some point $\boldsymbol{s}$ in the dual lattice $\Lambda^* = \{\boldsymbol{s} \in \mathbb{R}^{\ell+1} : \langle \boldsymbol{r}, \boldsymbol{s}\rangle \in \mathbb{Z}\}$ satisfies $0 < |\boldsymbol{s}|_\infty \ll R^{-1}$. Write $\boldsymbol{s} = g_t' u_{\boldsymbol{x}}'(\boldsymbol{q}, p)$ for some $p \in \mathbb{Z}$, $\boldsymbol{q} \in \mathbb{Z}^\ell$, where $g_t'$ and $u_{\boldsymbol{x}}'$ denote the inverse transposes of $g_t$ and $u_{\boldsymbol{x}}$, respectively. Then the inequality $|\boldsymbol{s}|_\infty \ll R^{-1}$ becomes

$$\begin{aligned} e^{-t/\ell}|\boldsymbol{q}|_\infty &\ll R^{-1} \\ e^t|\langle \boldsymbol{q}, \boldsymbol{x}\rangle + p| &\ll R^{-1} \end{aligned} \qquad \text{i.e.} \qquad \begin{aligned} |\boldsymbol{q}|_\infty &\ll \delta^{-1} \\ |\langle \boldsymbol{q}, \boldsymbol{x}\rangle + p| &\ll N^{-1}. \end{aligned}$$

Since $\delta \ge N^{-1/\tau}$ we get

$$|\langle \boldsymbol{q}, \boldsymbol{x}\rangle + p| \ll \delta^\tau \ll |\boldsymbol{q}|_\infty^{-\tau}. \tag{7}$$

Since $\tau > \tau_D(\boldsymbol{x})$, there are only finitely many pairs $(p, \boldsymbol{q})$ satisfying (7). Thus for all sufficiently large $N$, we have $\mathcal{D} \subseteq B_{\ell+1}(\mathbf{0}, R)$ and thus (6) holds. □

From this we can deduce the following consequence.

**Corollary 8.** *Let $\boldsymbol{x} \in \mathbb{R}^{d-1}$ be of dual Diophantine type $\tau_D(\boldsymbol{x}) < d$ and suppose that for all $\varepsilon > 0$, we have $\psi(q) \ge q^{-1/d-\varepsilon}$ for all $q$ sufficiently large. Then for any $k \ge 2$ and $\ell \in \mathbb{Z}$, we have*

$$\left|\left\{0 < q \le k^{j+\ell} : \|q\boldsymbol{x}\| < \psi(k^j)\right\}\right| \ll k^{j+\ell}\psi(k^j)^{d-1}$$

*for $j$ large enough.*

*Proof.* We show that for large enough $j$ we are in a situation where we can apply Lemma 7, with $N = k^{j+\ell}$ and $\delta = \psi(k^j)$. Since $\tau_D < d$ we can choose $\tau \in (\tau_D, d)$ and then for all large enough $j$,

$$N^{-1/\tau} = k^{-(j+\ell)/\tau} < \psi(k^j);$$

hence Lemma 7 applies. □

We are now ready for the last proof of this paper.

*Proof of Theorem 2(ii).* Let $\boldsymbol{x} \in \mathbb{R}^{d-1}$ be a point whose dual Diophantine type is strictly less than $d$, and let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a nonincreasing function such that $\sum_{q\in\mathbb{N}} \psi(q)^d$ diverges. Furthermore, assume that for every $\varepsilon > 0$, the inequality $1 > \psi(q) \ge q^{-1/d-\varepsilon}$ holds for all sufficiently large $q$. Then by Corollary 8, we satisfy all the parts of Lemma 6, so there exists $k \geq 2$ such that **(U)** holds. Thus by the argument given earlier, we can use [BDV06, Corollary 2] to conclude that almost every point on the line $\{\boldsymbol{x}\} \times \mathbb{R} \subseteq \mathbb{R}^d$ is $\psi$-approximable.

We now show that that the assumption that $1 > \psi(q) \geq q^{-1/d-\varepsilon}$ (for large enough $q$) can be made without losing generality. If $\psi(q) \geq 1$ for all $q$, then all points are $\psi$-approximable and the theorem is trivial, and if $\psi(q) < 1$ for some $q$, then by monotonicity $\psi(q) < 1$ for all $q$ sufficiently large. So we just need to show that the assumption $\psi(q) \geq q^{-1/d-\varepsilon}$ can be made without loss of generality. Let $\phi(q) = (q(\log(q))^2)^{-1/d}$ and define the function $\overline{\psi}(q) = \max\{\psi(q), \phi(q)\}$. Then $\overline{\psi}$ satisfies our assumptions, and therefore almost every point on $\{\boldsymbol{x}\} \times \mathbb{R}$ is $\overline{\psi}$-approximable. But

$$\sum_{\|q\boldsymbol{x}\| < \phi(q)} \phi(q) \leq \sum_{j\in\mathbb{N}} \phi(2^j) \left|\{0 < q \leq 2^{j+1} : \|q\boldsymbol{x}\| < \phi(2^j)\}\right| \overset{\text{Cor. } 8}{\ll} \sum_{j\in\mathbb{N}} 2^{j+1}\phi(2^j)^d,$$

which converges because $\sum_{q\in\mathbb{N}} \phi(q)^d$ does, and therefore almost every point on $\{\boldsymbol{x}\} \times \mathbb{R}$ is not $\phi$-approximable. But every $\overline{\psi}$-approximable point which is not $\phi$-approximable is $\psi$-approximable. Therefore, the set of $\psi$-approximable points on the line $\{\boldsymbol{x}\} \times \mathbb{R} \subseteq \mathbb{R}^d$ is of full measure, and the theorem is proved. □

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